An overview of Coupled Vortex Dynamics and the Hamiltonian Description

Aislinn Smith

January 2020

1 Abstract

My reading during this semester's directed reading program has mainly been to brief me on the topics in mathematical physics used to characterize fluid mechanics and plasma physics. I plan on applying what I have learned towards a senior thesis, so I also writing this to keep a strong record of the material I covered this semester. This presentation provides a specific example of ideal fluid flow and irrotational coupled vortices whose behavior can be described as a simple Hamiltonian system. I have also included a broad overview and definitions for Hamiltonians and Lagrangian systems as well as integrable Hamiltonians. The main background needed to understand this presentation is differential and integral calculus, vector calculus identities, and some basic understanding of continuum mechanics.

2 Introduction to Poisson Algebra and Legendre Transforms of a canonical Lagrangian system

The Lagrangian of a classically derived mechanical system (L(q,$\dot{q}$,t)) always satisfies the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

Which is derived under the principle of least action, using Calculus of Variations, and a Lagrangian functional. It is typically defined as the difference between total kinetic energy and total potential energy of a system.

The canonical coordinate are generalized position (q), generalized velocity ($\dot{q}$), and generalized momentum (p). Depending on the dimensions of the system, the coordinates may be vector valued. They may be indexed up to N, if there are N separate bodies in the system. In canonical coordinates, L(q,$\dot{q}$,t) often follows the form L = T($\dot{q}$,t) - U(q,t), such that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} = \frac{\partial V}{\partial q}$

Using a Legendre transform, the single 2nd order PDE for equation L(q,$\dot{q}$,t) can be transformed to an equation H(q,p,t) which can be solved as a coupled pair of first order PDEs, or ODEs if there is no time dependence, such that p (the generalized momenta) = $\frac{\partial L}{\partial \dot{q}}$

If the infinitesimal change of the Lagrangian is expressed via implicit differentiation, the Hamiltonian equation for the same mechanical system can be derived.:

$$dL = \frac{\partial L}{\partial q}dq + p\dot{q} + \frac{\partial L}{\partial t}dt$$

Via the product rule of integration:

$$\dot{q} = d(\dot{q}p) - \dot{q}dp$$

$$dL - d(\dot{q}p) = \frac{\partial L}{\partial q}dq - \dot{q}dp + \frac{\partial L}{\partial t}dt$$

Let $H(q,p,t) = L - \dot{q}p$

$$d(H) = -\frac{\partial L}{\partial q}dq + \dot{q}dp - \frac{\partial L}{\partial t}dt$$

Using the above equation, and taking into account the rules of implicit differentiation, it must be true that:

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p}$$
$$\frac{\partial H}{\partial p} = \dot{q}$$
$$\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$
$$\frac{\partial (p\dot{q})}{\partial t} = 0$$

3 Integrable Systems

An integrable system is a system of differential equations whose behavior (trajectory) can be determined by initial conditions and can be integrated (turned into a zeroth order derivative) via those integrals. These integrals are commonly referred to as first integrals or independent integrals. They describe the constants of motion for a Hamiltonian system.

By Liouvilles - Arnold Theorem of Integrable Hamiltonians (of n degrees of freedom, and of a 2n phase space):

If a Hamiltonian system is stationary (i.e. H(q,p,t) = H(q,p)), there are up to 2n-1 independent integrals of motion. An independent integral is of form $I(q, p, t_1) = I(q, p, t_2)$, such that $\{I, H\} = 0$. An *isolating* independent integral is special type of time invariant integral that reduces the dimension of a 2n phase space by one dimension. A system for which every independent integral is isolating, has a 1-d phase space trajectory (exists on a phase line or 1-d manifold). If there are n isolating integrals, the Hamiltonian will be integrable by quadrature. We say this set of independent integrals is in involution, meaning that they mutually commute with each other under Poisson brackets.

4 Poisson Equation and Constants of motion

$$f = f(q, p, t) \tag{1}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q} * \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial f}{\partial p} * \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial t} = \{H, f\} + \frac{\partial f}{\partial t} \tag{2}$$

$$\frac{\partial H}{\partial p} = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{3}$$

$$\frac{\partial H}{\partial q} = -\frac{\mathrm{d}p}{\mathrm{d}t} \tag{4}$$

If an integral ”f(q,p)” provides a constant of motion: $0 = \{H, f\}$ and,

$$\frac{\partial f}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{5}$$

An important characteristic of Hamiltonians is as follows:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q} * \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial H}{\partial p} * \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial H}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}t} * \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\mathrm{d}q}{\mathrm{d}t} * \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial H}{\partial t} = 0 + \frac{\partial H}{\partial t} \tag{6}$$

That is to say that if a Hamiltonian system has no implicit dependence $\frac{\partial H}{\partial t} = 0$, the Hamiltonian is completely invariant in time and forms an independent integral of which we can derive a constant of motion.

5 Navier Stokes and Ideal Flow

In order to derive the equation for the Hamiltonian describing a 2-body vortex, there are 3 main topics to go over:

- Idealized Flow Navier Stokes

- The stream function and velocity field

- The vortex equation

The Navier Stokes equation is an application of continuum mechanics and newton's law of conservation of momentum. The expression of body forces and and stress forces being equal to the material derivative of the fluid velocity field can be derived with some rigor. Since each particle moves with the velocity field; the material derivative is the Lagrangian expression of a fluid particles acceleration, expressed with Eulerian velocity field's (lab perspective) terms as follows.

$$\frac{Du(x,t)}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u \tag{7}$$

u(x,t) is a vector velocity field for fluid flow. The equations of idealized fluid flow assume constant density (ρ) of a particle/material volume and zero viscosity.

This is expressed in the continuity/ conservation of mass equation as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(x,t)dV = \int \frac{\partial \rho}{\partial t} + \nabla \cdot \rho(u(x,t)) = 0 \tag{8}$$

When flow is incompressible, the density of each particle is constant in time, but different particles may have different density. So it implies $\frac{\partial \rho}{\partial t} = 0$

$$\nabla \cdot (u(x,t)) = 0 \tag{9}$$

Using the conservation of mass in the Newtonian/System reference frame, we can integrate the time derivative density and velocity function to the body and contact forces on a material volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(x,t)u(x,t)dV = \int \rho(x,t)gdV + \int f(x,n,t) \cdot \hat{n}dA = \int \rho(x,t)gdV + \int f(x,n,t) \cdot \hat{n}dV \tag{10}$$

Since this material volume may have a time dependence for the area element expressed as $\vec{v_b}$ (deformation), the Reynolds transport theorem can be applied. Which is another expression of the material derivative. $\vec{v_r}$ is the velocity of the particles with respect to the material volume.

$$\int \rho(x,t) g dV + \int f(x,n,t) \cdot \hat{n} dV = \int \frac{\partial \rho(x,t) u(x,t)}{\partial t} dV + \int \rho * u(x,t)(v_b \cdot \hat{n}) dA + \int \rho * u(x,t)(v_r \cdot \hat{n}) dA$$

For an in-compressible fluid with no effect of gravity, this is integral equivalent to setting the material derivative to the below expression. Typically there is a term $v\nabla^2 u$, which is the expression of the disapative behavior of viscosity, which is alike in form to the heat equation. However, with idealized flow assumptions we remove this term as it actively removes the consistency of w in time. For ideal flow, the material derivative requires that there is no implicit reliance on time for the velocity field (Eulerian POV).

$$\frac{Du(x,t)}{Dt} = -\frac{1}{\rho}\nabla P + v\nabla^2 u \tag{11}$$

The vorticity (ω) of the velocity field is defined as the curl of the velocity field at a point. If we take the curl of the conservation of momentum equation (Navier Stokes), we yield this:

$$\nabla \times \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = 0 \implies \frac{\partial \omega}{\partial t} + \nabla \times (\omega \times u) = \nu \nabla^2 \omega \tag{12}$$

The material derivative for the vorticity, takes the same form as the the material derivative of the velocity field (i.e a partial time derivative and an advection term)

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u + \nu \nabla^2 \omega - \nabla \times (\omega \times u). \tag{13}$$

Since in an ideal flow viscocity is zero, that term goes to zero. Since $\nabla \times (\omega \times u) = (\nabla \cdot u)\omega - (\nabla \cdot \omega) u$. We can set that term to zero as well because we have acknowledged for ideal flow that div(w) and div(u) = 0.

So, for ideal flow:

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u \tag{14}$$

However as a further simplification, often ideal flow can be approximated to be confined to a 2d plane, if the depth of the fluid is of negligible width compared to the surface area. This approximation often works for large bodies of fluid with extreme surface area to depth ratios like the ocean or the atmosphere.

If we confine a fluid's flow velocity to an "x,y" plane, the curl(u) will always be orthoganol to any point vortex, ie, in the z direction.

$<\mathrm{w}_x, w_y, w_z > \cdot < \frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, 0 >=< 0, 0, \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} > \cdot < \frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, 0 >= 0 = \frac{Dw}{Dt}$

This means that along a streamline (a path that is parallel to the vectors of fluid flow), if there is zero vorticity at t_o there will be zero vorticity along that stream line for all points in time.

6 Stream Function and Velocity Potential

The stream function (ψ) is defined for any in-compressible (divergence free) fluid as $u = (\nabla \times \psi)$ The stream function will be constant along the stream lines during steady state flow. The difference in the stream function at two different points in space, will show the volumetric flow between those two points.

For irrotational flow, the velocity field u(x,t) can be written at the gradient of a velocity potential field.

$$u = \nabla\ \phi(x,t) \quad \Longrightarrow \quad \nabla \times \nabla\ \phi(x,t) = 0 \tag{15}$$

For incompressible flow, u(x,t) can be expressed as the potential of a solenoidal vector field, the stream function.

volume flux parallel to the normal vector of a 2-d contour $= \int u_x \delta y + u_y \delta x = \int \frac{\partial \psi}{\partial y} \delta y + \frac{\partial \psi}{\partial x} \delta x = \psi =$ a constant

So if the curl of a divergence free 2-D velocity field ($\delta \psi = 0$) starts out as zero, and is an ideal fluid:

$$0 = w_z = \nabla\times u = \nabla\times\nabla\times\psi = -\frac{\partial^2\psi}{\partial y\partial y}+\frac{\partial^2\psi}{\partial x\partial x} = -\nabla^2\psi+\psi(\nabla\cdot\psi) = -\nabla^2\psi = 0 \quad (16)$$

As a side note, because the gradient of velocity potential equals u and u = $< \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial y} >$, $\nabla\psi\cdot\nabla\phi = 0$. This means that the gradient of the velocity potential and the gradient of the stream function are orthogonal. Lastly, Since $\nabla\cdot u = 0$, the velocity potential function also satisfies its own Poisson equation $\nabla^2\phi = 0$.

7 Green's Functions and Solving the Poisson eqn for ideal irrotational point vortices on a 2-D plane:

An irrotational point vortex, in non-mathematical terms, is any vortex for which the circulation is zero everywhere except for a point source in the center of a vortex.

The Poisson equation, which is typically the Laplacian of a function set to zero, can also be set equal to a dirac delta function of form $-\tau\delta(x-x_i)\delta(y-y_i)$ where the point (x_i, y_i) is marked as the coordinate of the center of the vortex, and τ is the strength of the ideal vortex circulation.

We can use Green's functions or separation of variables to solve for an elementary and unbounded solution for this Poisson equation that is only a function of the position vector on a 2-d plane.

$$\nabla^2\psi_i = -\tau\delta(x-x_i)\delta(y-y_i) \quad (17)$$

$$\psi_i = \int \frac{\tau\delta(x-x_i)\delta(y-y_i)}{r}dV' = -\frac{\tau_i}{2\pi}\ln\sqrt{(x-x_i)^2+(y-y_i)^2} = C \text{ a constant} \quad (18)$$

C is a constant because for a divergence free flow, $\delta\psi = 0$

Given that we have already defined the relationship between the velocity field in a divergence free fluid and the stream function as a vector.:

$$u =< \frac{\partial \psi_i}{\partial y}, -\frac{\partial \psi_i}{\partial x} > \tag{19}$$

or in equivalent form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla \psi_i(x, y) \tag{20}$$

Now, keeping in mind that the stream function (the contours or streamlines of which are tangent to fluid flow) also contains in it the center coordinates of the irrotational vortex, we can characterize the behavior of two nearby vortices in the same fluid. Since the Poisson operator is a linear operator, multiple stream function solution from multiple Poisson equations (produced by multiple vortices) can be added to form a stream function solution for an entire field of point vortices.

Since the flow of all the particles follow the path of the streamlines, a nearby vortex's center will also fall in line with the stream lines produced by a nearby vortex, but the stream function it produces will not affect it's own trajectory. This is similar to how a charge can be acted upon by nearby charges' electric fields, but never by it's own electric field.

If we consider another irrotational vortex with it's center located at (x_j, y_j), it will have a nearly identical stream function ψ_j

$$\psi_j = -\frac{\tau_j}{2\pi} \ln \sqrt{(x - x_j)^2 + (y - y_j)^2} \tag{21}$$

If we apply the stream function from vortex i onto vortex j with the following matrix, the following vector valued ODE is the result.

$$\frac{d}{dt}\begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla \psi_i(x_j, y_j) = \frac{\tau_i}{2\pi} \begin{bmatrix} \frac{-(y_j - y_i)}{|r_j - r_i|^2} \\ \frac{(x_j - x_i)}{|r_j - r_i|^2} \end{bmatrix} \tag{22}$$

$$\dot{y}_j \tau_j 2\pi = \tau_j \tau_i \frac{(x_j - x_i)}{|r_j - r_i|^2} \tag{23}$$

$$\dot{x}_j \tau_j 2\pi = -\tau_j \tau_i \frac{(y_j - y_i)}{|r_j - r_i|^2} \tag{24}$$

If we apply the stream function ψ_j upon the center of the vortex i, in the same manner as above, we get another set of similar equations:

$$\dot{y}_i \tau_i 2\pi = \tau_j \tau_i \frac{(x_i - x_j)}{|r_j - r_i|^2} \tag{25}$$

$$\dot{x}_i \tau_i 2\pi = -\tau_j \tau_i \frac{(y_i - y_j)}{|r_j - r_i|^2} \tag{26}$$

We essentially find 4 ODE's describing the motion of the two vortices with respect to their separation.

An important characteristic of these equations is that the velocity vector for a particular particle (or the center of a vortex as we have demonstrated from above) in a stream line, is the result of a spacial derivative of a stream function.

Through an constructive process a Hamiltonian description of this system can be uncovered for TWO OR MORE coupled vortices. (Note: I have been using the i,j notation index for a 2 vortex system, but the following equation can be expanded for any number of vortices)

$$H(x,y) = -\sum_{i>j} \gamma_i \gamma_j \ln(|\vec{r_i} - \vec{r_j}|) \tag{27}$$

where the non-canonical coordinates x and y (for each of the N vortices) are the non-canonical coordinates, which in this case means neither of the coordinates stand for generalized momentum. Despite this, the equation above still satisfies the requirement of a Hamiltonian system. I will discuss this more in the next section.

8 Hamiltonian Description of Coupled Vortices

If the partial derivative of H is taken with respect to x (one of the canonical coordinates), as well as the partial derivative of H with respect to y (the other canonical coordinate), we get the following two equations, which show that this equation 28 meets the requirements of a Hamiltonian system.

(*Note, the brackets represent Poisson brackets*)

$$\dot{x}_i = \{x_i, H\} = \frac{1}{\gamma_i}\frac{\partial H}{\partial y_i} \quad \text{which is equivalent to the requirement that} \quad \dot{q} = \{q, H\} = \frac{\partial H}{\partial p} \tag{28}$$

$$\dot{y}_i = \{y_i, H\} = -\frac{1}{\gamma_i}\frac{\partial H}{\partial x_i} \quad \text{which is equivalent to the requirement that} \quad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial q} \tag{29}$$

These are the same relations given by the coupled vortex system as function of multiple stream functions, so this Hamiltonian is physically sound.

This also guarantees that H(x,y) will commute with itself, such that $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}$. As we can see, the above Hamiltonian has no implicit time dependence, and therefore it has no time dependence at all.

8.1 2-body Hamiltonian Solution

The Non-canonical coordinates of the Hamiltonian describing the motion of a simplified (point vortexes do not merge) 2-body vortex system is a such:

$$x' \equiv q = x_1 - x_2 \tag{30}$$

$$y' \equiv p = y_1 - y_2 \tag{31}$$

By assigning x' and y' to be the seperation of the 2 vortices, we have reduced a 2 body Hamiltonian to 1 body. Therefore, the phase space of this system has dimension

2, meaning that this system will be integrable if there is at least one independent time invariant integral I(x,y)

$$H(x,y) = \gamma_1\gamma_2 * ln[\sqrt[2]{(x_1 - x_2)^2 + (y_1 - y_2)^2)}] \quad (32)$$

$$H(x',y') = \gamma_1\gamma_2 * ln[\sqrt[2]{(x')^2 + (y')^2)}] \quad (33)$$

Since this is a hamiltonian that takes up a 2-d manifold, and has at least one constant of motion H(x',y'), we can already establish that the system is integrable and the equations of motion can be solved to find a 1D trajectory of motion, with coordinates (x',y').

Besides the Hamiltonian itself, there are 3 other integrals of motion that have no implicit time dependence and commute with H(x',y') under Poisson brackets.

$$P_x = \gamma_1 * y_1 + \gamma_2 * y_2 \implies \{P_x, H\} = 0$$

$$P_y = \gamma_1 * x_1 + \gamma_2 * x_2 \implies \{P_y, H\} = 0$$

$$J = -\frac{1}{2}(\gamma_1(x_1^2 + y_1^2) + \gamma_2(x_2^2 + y_2^2)) \implies \{J, H\} = 0$$

So all 3 of these integrals plus the Hamiltonian itself are conserved quantities, however, only 3 integrals are in involution (meaning they all mutually commute):

$$\{Px + Py, H\} = 0 = \{Px + Py, J\} = 0 = \{J, H\} = 0$$

This is why only a system of up to 3 vortices is integrable. Above that number there are not enough constants of motion, and the motion is described as chaotic.

Since $\frac{\partial H}{\partial t} = 0$, we know that the phase space trajectory with axis $x' = (x_1 - x_2)$ and $y' = (y_1 - y_2)$ will occupy only a 1-d manifold. As mentioned in the previous section $\frac{\partial H}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}$, meaning that the magnitude of vector $|\vec{r}| = |\vec{r_1} - \vec{r_2}| = R$ (a constant).

$$c = -\gamma_1 * \gamma_2 \tag{34}$$

Using equations 29 and 30, we can construct a system of first order differential equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\gamma_2 \frac{y_1 - y_1}{R^2} \tag{35}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\gamma_1 \frac{y_2 - y_1}{R^2} \tag{36}$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \gamma_2 \frac{x_1 - x_2}{R^2} \tag{37}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = \gamma_1 \frac{x_2 - x_1}{R^2} \tag{38}$$

By subtracting equation 37 from equation 36, and subtracting equation 39 from 38, we can simplify the problem to one body.

$$\frac{\mathrm{d}x_1 - x_2}{\mathrm{d}t} = \frac{\mathrm{d}x'}{\mathrm{d}t} = (-\gamma_2 - \gamma_1) * \frac{y_1 - y_2}{R^2} = (-\gamma_2 - \gamma_1) * \frac{y'}{R^2} \tag{39}$$

$$\frac{\mathrm{d}y_1 - y_2}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\mathrm{d}t} = (\gamma_2 + \gamma_1) * \frac{x_1 - x_2}{R^2} = (\gamma_2 + \gamma_1) * \frac{x'}{R^2} \tag{40}$$

We will also define ω as $\frac{(\gamma_1 + \gamma_2)}{R^2}$

Using a simple substitution of equation 40 into equation 41, and visa versa, we will have two second order ODE's with complex solutions:

$$\frac{\mathrm{d}^2 x'}{\mathrm{d}t^2} = -\omega^2 x' \tag{41}$$

$$\frac{\mathrm{d}^2 y'}{\mathrm{d}t^2} = -\omega^2 y' \tag{42}$$

Once the intitial condition that $x'^2 + y'^2 = R$ is satisfied, the following simple solutions result.

This tells us that the two vortices orbit around each other at a frequency inversely proportionally to their separation. It is important to note that this solution results when the circulation strength constants γ_1 and γ_2 have the same sign, meaning the vortices circular stream lines point in the same direction. A similar phenomenon is called the Fuhiwara effect.This is seen when hurricanes with the same direction of circulation briefly rotate around each other before colliding.

$$x_1 - x_2 = R * \sin\Big(\frac{\omega}{R^2}(t - t_o)\Big) \tag{43}$$

$$y_1 - y_2 = R * \cos\Big(\frac{\omega}{R^2}(t - t_o)\Big) \tag{44}$$

9 Conclusion

This project has helped to ground me in some of the central ideas of continuum fluid mechanics and Hamiltonian systems. There are some lingering physical questions I wish to explore later, such as; the mathematical relations between coupled fluid vortices and interacting electromagnetic dipoles. I also would like learn more about ideal flow conditions and where irrotational vortices from point ”sinks” appear in nature. Because of this project I now have my sights set on some more advanced mathematical topics. These topics include symplectic geometry, Lagrangian manifolds, Lie algebra, and Hamiltonian-Jacobi transformations, and are important to the branch of applied mathematics I plan to study.

References

Morrison, Philip. (1981). Hamiltonian field description of two-dimensional vortex fluids and guiding center plasmas. 10.2172/6351319.

http://www.astro.yale.edu/vdbosch/lecture4.pdf
https://www.damtp.cam.ac.uk/user/tong/dynamics/four.pdf
https://www-users.math.umn.edu/~olver/t_/poisson.pdf
http://web.mit.edu/13.021/demos/lectures/lecture2.pdf-ReferenceFrames
http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/Fluidhtml.html
https://www3.nd.edu/~gtryggva/CFDCourse/2011-Lecture-30.pdf